Summary

A useful result for manipulating algebraic expression, Matrix Inversion Lemma will be the focus of this note, where a full derivation is provided.

1 Matrix Inversion Lemma

Lemma 1: For square and invertible $\boldsymbol{X}, \boldsymbol{Y}$ and conformable $\boldsymbol{U}, \boldsymbol{V}$:

$$(\boldsymbol{X}+\boldsymbol{U}\boldsymbol{Y}\boldsymbol{V})^{-1} = \boldsymbol{X}^{-1} - \boldsymbol{X}^{-1}\boldsymbol{U}[\boldsymbol{Y}^{-1}+\boldsymbol{V}\boldsymbol{X}^{-1}\boldsymbol{U}]^{-1}\boldsymbol{V}\boldsymbol{X}^{-1}\ , \tag{1}$$

if the inverse exists.

Proof: by multiplying the left-hand side and right-hand side.

$$\begin{aligned}
&[(\boldsymbol{X}+\boldsymbol{U}\boldsymbol{Y}\boldsymbol{V})]\left[\boldsymbol{X}^{-1} - \boldsymbol{X}^{-1}\boldsymbol{U}[\boldsymbol{Y}^{-1}+\boldsymbol{V}\boldsymbol{X}^{-1}\boldsymbol{U}]^{-1}\boldsymbol{V}\boldsymbol{X}^{-1}\right] \\
&\quad = \boldsymbol{X}\boldsymbol{X}^{-1} - \boldsymbol{X}\boldsymbol{X}^{-1}\boldsymbol{U}[\boldsymbol{Y}^{-1}+\boldsymbol{V}\boldsymbol{X}^{-1}\boldsymbol{U}]^{-1}\boldsymbol{V}\boldsymbol{X}^{-1} + \boldsymbol{U}\boldsymbol{Y}\boldsymbol{V}\boldsymbol{X}^{-1} \\
&\qquad\qquad - \boldsymbol{U}\boldsymbol{Y}\boldsymbol{V}\boldsymbol{X}^{-1}\boldsymbol{U}[\boldsymbol{Y}^{-1}+\boldsymbol{V}\boldsymbol{X}^{-1}\boldsymbol{U}]^{-1}\boldsymbol{V}\boldsymbol{X}^{-1} \\
&\quad = \boldsymbol{I} + \boldsymbol{U}\boldsymbol{Y}\boldsymbol{V}\boldsymbol{X}^{-1} \\
&\qquad\qquad - [\boldsymbol{U}+\boldsymbol{U}\boldsymbol{Y}\boldsymbol{V}\boldsymbol{X}^{-1}\boldsymbol{U}][\boldsymbol{Y}^{-1}+\boldsymbol{V}\boldsymbol{X}^{-1}\boldsymbol{U}]^{-1}\boldsymbol{V}\boldsymbol{X}^{-1} \\
&\quad = \boldsymbol{I} + \boldsymbol{U}\boldsymbol{Y}\boldsymbol{V}\boldsymbol{X}^{-1} \\
&\qquad\qquad - \boldsymbol{U}\boldsymbol{Y}[\boldsymbol{Y}^{-1}+\boldsymbol{V}\boldsymbol{X}^{-1}\boldsymbol{U}][\boldsymbol{Y}^{-1}+\boldsymbol{V}\boldsymbol{X}^{-1}\boldsymbol{U}]^{-1}\boldsymbol{V}\boldsymbol{X}^{-1} \\
&\quad = \boldsymbol{I} + \boldsymbol{U}\boldsymbol{Y}\boldsymbol{V}\boldsymbol{X}^{-1} - \boldsymbol{U}\boldsymbol{Y}\boldsymbol{V}\boldsymbol{X}^{-1} = \boldsymbol{I}
\end{aligned} \tag{2}$$

This completes the proof. □